Consistency and Identification

adapted from Arne Gabrielsen, "Consistency and Identifiability," *Journal of Econometrics* 8 (1978) pp 261-263.

Definition If complete knowledge of the distribution of the observed variables gives enough information to get exact knowledge of a parameter, the parameter is **identified** in the model.

Definition Let $\hat{\theta}_n$ be an estimator for the parameter θ, based on n observations. $\hat{\theta}_n$ is a **consistent** estimator if it converges in probability to the true value of θ, independent of any particular value of θ, or more precisely, if for any real $\varepsilon > 0$,

$$\lim_{n\to\infty} \Pr\left(\left|\hat{\theta}_n \to \theta\right| < \varepsilon\right) = 1$$

Proposition The existence of a consistent estimator for θ implies that it is identified. (demonstrated on p. 262)

Proposition There may not be a consistent estimator for θ even if it is identified.

Proposition 2 can be shown by example. Consider the model:

$$y_i = \beta r^i + u_i \text{ for i=1,2,...,n}$$

- where the u_i's are independently drawn from a standard normal distribution, $r \in (0,1)$ is a known scalar constant, and β is a nonnegative parameter to be estimated. Here r taken to the power of i is the independent variable. Let x be the vector $[r^i]$.

β is identified: since $Ey_i = \beta r^i$, if we knew the true distribution of y_i, we could immediately calculate β by $\beta = \frac{Ey_i}{r^i}$.

But no consistent estimator for β exists.

Suppose for example we attempt to estimate β by OLS, which is in this case the maximum likelihood estimator.

$$\hat{\beta}_{OLS} = \left(x'x\right)^{-1}x'y = \frac{\sum_{i=1}^{n} r^i y_i}{\sum_{i=1}^{n} r^{2i}}.$$

$\hat{\beta}_{OLS}$ is normally distributed with expectation β (it is unbiased) and variance

$$\sigma_n^2 = (x'x)^{-1} = \frac{1}{\sum_{i=1}^{n} r^{2i}} = \frac{1}{r^2+r^4+...+r^{2n}} = \frac{1}{r^2(1+r^2+...+r^{2n})} = \frac{1}{r^2}\left(\frac{1}{1-r^2} - \frac{r^{2n}}{1-r^2}\right)$$

$$= \frac{r^2(1-r^{2n})}{1-r^2}$$

As $n \to \infty$, σ_n^2 declines monotonically to $\frac{1-r^2}{r^2}$. Since this variance does not collapse to zero, $\hat{\beta}_{OLS}$ is not consistent.

It can be shown by the Neyman-Pearson Lemma that no consistent estimator for β exists. (See p. 263)